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Non-Linear Analysis on Transient Heat Transfer in Annular Fin

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Abstract:

The present paper deals with heat transfer analysis of an annular fin with variable thermal properties. The fin is under transient heat conduction. A constant temperature T_b is applied at the inner circular boundary ($r = r_i$). The top surface ($z = 2a$) of the fin dissipates heat to the surroundings by convection. The outer circular boundary ($r = r_0$) and the lower surface ($z = 0$) are thermally insulated. Initially the fin is kept at constant temperature. The governing nonlinear differential equation is solved by finite difference method. The convergence and stability analysis of finite difference scheme has been done by fundamental theorems of numerical analysis. The results of temperature has been computed numerically, illustrated graphically and interpreted technically.

Keywords — **Annular fin, Transient heat conduction, Non-linear boundary value problem, Finite difference method**.

I. INTRODUCTION

In a variety of engineering applications, extended surfaces are frequently adopted to enhance the rate of heat dissipation between the system and the surroundings. The heat transfer mechanism of fin is to conduct heat from heat source to the fin surface via conduction, and then dissipate heat to the surrounding fluid via convection, radiation, or simultaneous convection–radiation. In order to design a practical fin, it is necessary to realize a fin's dynamic temperature response. In the case of constant thermal conductivity, the analytical solution can be easily obtained. In fact, a considerable amount of research has been conducted into the variable thermal parameters which associated with fins operating in practical situations. In such case, the governing equation of fin will be nonlinear and a numerical treatment-with suitable algorithms.

Roy Chaudhari [12] wrote a note on Quasi Static thermal deflection of a thin clamped circular plate due to ramp type heating of a concentric circular region of the upper face. Hsin-Ping Chu et al [4] outlined the differential transformation technique and then procedures for transforming and discretizing the governing equations as well as the boundary conditions are given in two numerical examples. Maerefat M. et al [7] have applied hybrid differential transform and finite difference method to solve 2D transient nonlinear straight annular fin equation. Huan-Sen Peng et al [5] have utilized a hybrid numerical technique which combines the differential transformation and finite difference method to investigate the annular fin with temperature-dependent thermal conductivity. Aksoy I. G. [2] compared results of the homotopy analysis method are with numerical results of the finite difference method for the thermal analysis of annular fins with temperature-dependent thermal properties. Pranab Kanti Roy et al. [11] studied application of homotopy perturbation method for a conductive–radiative fin with temperature dependent thermal conductivity and surface emissivity. Recently, M. G. Sobamowo [6] analyses the optimum design dimensions and performance of convective-radiative cooling fin subjected to magnetic field are presented using fnite element method.

Numerical methods are useful for solving partial differential equations of science and engineering when such problems cannot be handled by the exact analysis because of nonlinearities, complex geometries, and complicated boundary conditions. The development of the high-speed digital computers significantly enhanced the use of numerical methods. Theoretical results have been obtained during the last five decades regarding the accuracy, stability and convergence of the finite difference method for partial differential equations.

Cherruault Y. et al [3] have solved nonlinear partial differential equations in Biochemistry by Crank- Nicolson technique and proved the convergence of the same. Akil J. Harfash [1] developed a compact finite difference scheme to the three-dimensional microscale heat transport equation. and proved to be unconditionally stable with respect to initial values. Marco Picasso et al [8] applied an adaptive algorithm for the Crank Nicolson scheme to a time-dependent convection-diffusion problem.

In this paper an attempt is made to solve the nonlinear governing differential equation by finite difference method. The convergence and stability analysis has been done to validate obtained results. The temperature and thermal stresses computed numerically, illustrated graphically and interpreted technically.

II. PROBLEM FORMULATION

Consider a straight annular fin of thickness $2a$ occupying space D defined by $r_i \le r \le r_0$, $0 \le z \le 2a$ as shown in Figure 1. Initially the plate is kept at constant temperature T_∞ . At the upper surface of the fin convection due to dissipation takes place. The lower boundary surface at $z = 0$ and outer circular boundary at $r = r_0$ are kept insulated.

The governing equation, initial and boundary conditions for temperature field [7] consist of:

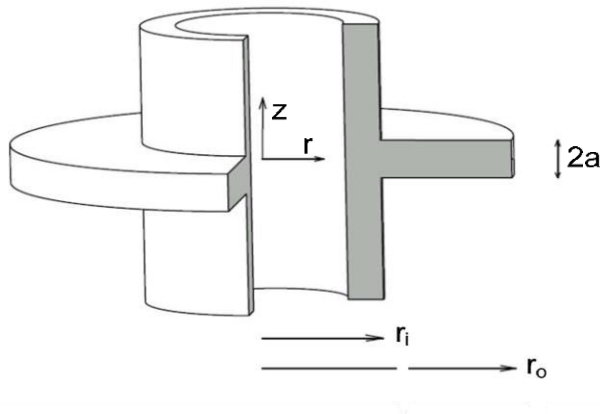

Fig. 1 The geometry of heat conduction problem

1. the nonlinear governing equation

$$\frac{\partial}{\partial r}\left(k(T)\frac{\partial T}{\partial r}\right)+\frac{k(T)}{r}\frac{\partial T}{\partial r}+\frac{\partial}{\partial z}\left(k(T)\frac{\partial T}{\partial z}\right)=\rho c_p(T)\frac{\partial T}{\partial t} \tag{1}$$

2. the initial condition

$$T = T_\infty \qquad \text{at } t=0, r_i \le r \le r_0, 0 \le z \le 2a \tag{2}$$

3. the boundary conditions

$$T = T_b \qquad \text{at } r = r_i, t > 0 , \tag{3}$$

$$\frac{\partial T}{\partial r} = 0 \qquad \text{at } r = r_0, t > 0 , \tag{4}$$

$$\frac{\partial T}{\partial z} = 0 \qquad \text{at } z = 0,\ t > 0 , \tag{5}$$

$$-k(T)\frac{\partial T}{\partial z} = h(T - T_\infty) \qquad \text{at } z = 2a,\ t > 0 \tag{6}$$

The thermal conductivity of any metal depends upon the temperature. Following [10] the thermal conductivity k^{n+1} at the time level n+1 is expresses in term of that at the time level n in the form

$$k^{n+1} \cong k^n + \left(\frac{\partial k}{\partial t}\right)^n \Delta t$$

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Replacing the time derivative and if the thermal conductivity varies linearly with temperature one gets

$$k^{n+1} \cong k^n \left[1 + \beta(T^n - T^{n-1})\right] \tag{7}$$

Similar expression can be written for the specific heat

$$c_p^{n+1} \cong c_p^n \left[1 + \gamma(T^n - T^{n-1})\right] \tag{8}$$

Equations (1) to (8) constitute the mathematical formulation of the problem.

III. MATHEMATICAL SOLUTION

A complete finite difference model is proposed for heat transfer analysis is given by

One can divide the $r,\ z,\ t$ domain into small intervals $\Delta r,\ \Delta z,\ \Delta t$ such that

$r = i\Delta r \quad i = 0,1,.....N \ \ (N\Delta r = r_0)$

$z = j\Delta z \quad j = 0,1,2,........\mathrm{M} \ \ (M\Delta z = 2a)$

$t = n\Delta t \quad n = 0,1,2,........$

The temperature at the nodal point $(i\Delta r,\ \mathrm{j}\Delta z)$ at the time $n.\Delta t$ is denoted by $T(i.\Delta r,\ j.\Delta z) = T_{i,\,j}^{n}$

The Crank Nicolson finite difference representation [10] of the two dimensional nonlinear heat equation (1) is given by

$$\begin{aligned}
\rho(c_p)_{i,j}\frac{T_{i,j}^{n+1}-T_{i,j}^{n}}{\Delta t} &= \frac{1}{2}\left[k_{i-1/2,j}\frac{T_{i-1,j}^{n+1}-T_{i,j}^{n+1}}{(\Delta r)^2}+k_{i+1/2,j}\frac{T_{i+1,j}^{n+1}-T_{i,j}^{n+1}}{(\Delta r)^2}\right]\\
&+\frac{1}{2}\left[k_{i-1/2,j}\frac{T_{i-1,j}^{n}-T_{i,j}^{n}}{(\Delta r)^2}+k_{i+1/2,j}\frac{T_{i+1,j}^{n}-T_{i,j}^{n}}{(\Delta r)^2}\right]\\
&+\frac{1}{2}\frac{k_{i,j}}{i\Delta r}\left[\frac{T_{i+1,j}^{n+1}-T_{i-1,j}^{n+1}}{2\Delta r}+\frac{T_{i+1,j}^{n}-T_{i-1,j}^{n}}{2\Delta r}\right]\\
&+\frac{1}{2}\left[k_{i,j-1/2}\frac{T_{i,j-1}^{n+1}-T_{i,j}^{n+1}}{(\Delta z)^2}+k_{i,j+1/2}\frac{T_{i,j+1}^{n+1}-T_{i,j}^{n+1}}{(\Delta z)^2}\right]\\
&+\frac{1}{2}\left[k_{i,j-1/2}\frac{T_{i,j-1}^{n}-T_{i,j}^{n}}{(\Delta z)^2}+k_{i,j+1/2}\frac{T_{i,j+1}^{n}-T_{i,j}^{n}}{(\Delta z)^2}\right]
\end{aligned} \tag{9}$$

The subscript $i \pm 1/2$ for the thermal conductivity denotes that a mean value of thermal conductivity between the nodal points $i \pm 1$ and i.

Solving (9) for $T_{i,j}^{n+1}$ and setting square grids $\Delta r = \Delta z$ one gets the recursive relation,

$$T_{i,j}^{n+1} = A_{ij}T_{i-1,j}^{n+1} + B_{ij}T_{i+1,j}^{n+1} + \mathrm{C}_{ij}T_{i,j-1}^{n+1} + D_{ij}T_{i,j+1}^{n+1} - E_{ij}T_{i,j}^{n+1} + b_{ij} \tag{10}$$

where the coefficients are given by

$$A_{ij} = \frac{\Delta t}{2(\Delta r)^2\rho(c_p)_{i,j}}\left[k_{i-1/2,j} - \frac{k_{i,j}}{2i}\right]$$

$$B_{ij} = \frac{\Delta t}{2(\Delta r)^2\rho(c_p)_{i,j}}\left[k_{i+1/2,j} + \frac{k_{i,j}}{2i}\right]$$

$$C_{ij} = \frac{k_{i,j-1/2}\Delta t}{2(\Delta z)^2\rho(c_p)_{i,j}}$$

$$D_{ij} = \frac{k_{i,j+1/2}\Delta t}{2(\Delta z)^2\rho(c_p)_{i,j}}$$

$$E_{ij} = A_{ij} + B_{ij} + C_{ij} + D_{ij}$$

$$b_{ij} = A_{ij}T_{i-1,j}^{n} + B_{ij}T_{i+1,j}^{n} + \mathrm{C}_{ij}T_{i,j-1}^{n} + D_{ij}T_{i,j+1}^{n} + (1 - E_{ij})T_{i,j}^{n} \tag{11}$$

The finite difference approximation for initial and boundary conditions,

Initially at $t=0$

$$T_{i,j}^{0}=T_{\infty} \tag{12}$$

At inner circular boundary ($r=r_i$)

$$T_{i,j}^{n}=T_{b} \tag{13}$$

At outer circular boundary ($r=r_0$)

$$T_{i+1,j}^{n}=T_{i-1,j}^{n} \tag{14}$$

At lower surface ($z=0$)

$$T_{i,j-1}^{n}=T_{i,j+1}^{n} \tag{15}$$

The upper surface ($z=2a$)

$$T_{i,j+1}^{n}=T_{i,j-1}^{n}+\frac{2h\Delta z}{k(T)}\left(T_{\infty}-T_{i,j}^{n}\right) \tag{16}$$

Equation (12) gives the initial value of temperature T at each grid point of the plate (at $t=0$). Assuming that the coefficients $A_{ij},B_{ij},C_{ij},D_{ij},b_{ij}$ are known for each iteration equation (10) with the boundary conditions (13) to (16) gives a set of linear equations, one can apply the successive over relaxation method to solve these equations. The recursive relation is given by,

$$T_{i,j}^{n+1}=(1-\omega)T_{i,j}^{n}+\omega\left[A_{ij}T_{i-1,j}^{n}+B_{ij}T_{i+1,j}^{n}+C_{ij}T_{i,j-1}^{n}+D_{ij}T_{i,j+1}^{n}-E_{ij}T_{i,j}^{n}+b_{ij}\right] \tag{17}$$

where relaxation factor is ω lies between 1 and 2

The truncation error is of order $O\left[(\Delta t)^2,(\Delta r)^2,(\Delta z)^2\right]$, the scheme is unconditionally stable and convergent by using following theorem due to Lax [13].

Theorem 1: A consistent difference scheme for a well posed linear initial boundary value problem is convergent if and only if it is stable.

IV. RESULTS AND DISCUSSION

The simultaneous equations formed by (12) to (17) are solved using MATLAB programming. The dimensions and material properties are as follows

A. Dimensions

Inner radius of annular fin $r_i=0.2m,$

Outer radius of annular fin $r_0=0.5m,$

Height of annular fin $2a=0.1m$.

B. Material properties

The numerical calculation has been carried out for an Aluminum (Pure) fin with the material properties as,

Thermal conductivity $k=204.2\ W/mK$,

Specific heat $c_p=896J/KgK$,

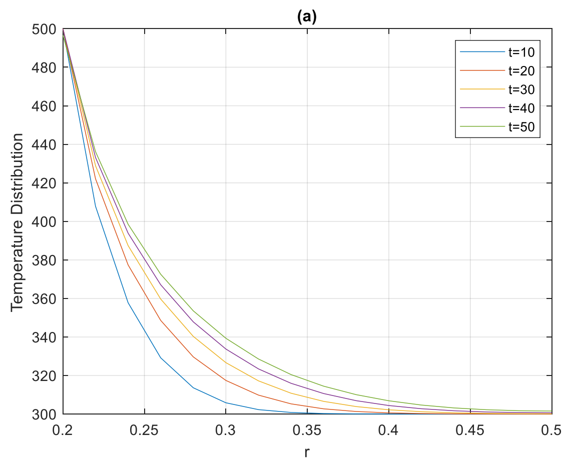
Density $\rho=2707Kg/m^3$,

Coefficient of convection $h=10$,

Temperature of surrounding media $T_{\infty}=300K$,

Temperature at inner circular radius $T_b=500K$.

Temperature in radial and axial direction at grid points equally spaced with $\Delta r=\Delta z=0.02$ meters and at time t = 500x0.1=50 seconds are determined when the thermal properties are independent ($\beta=0,\gamma=0$) and dependent ($\beta\neq 0,\gamma\neq 0$) on temperature. The plots along radial and axial directions are given by



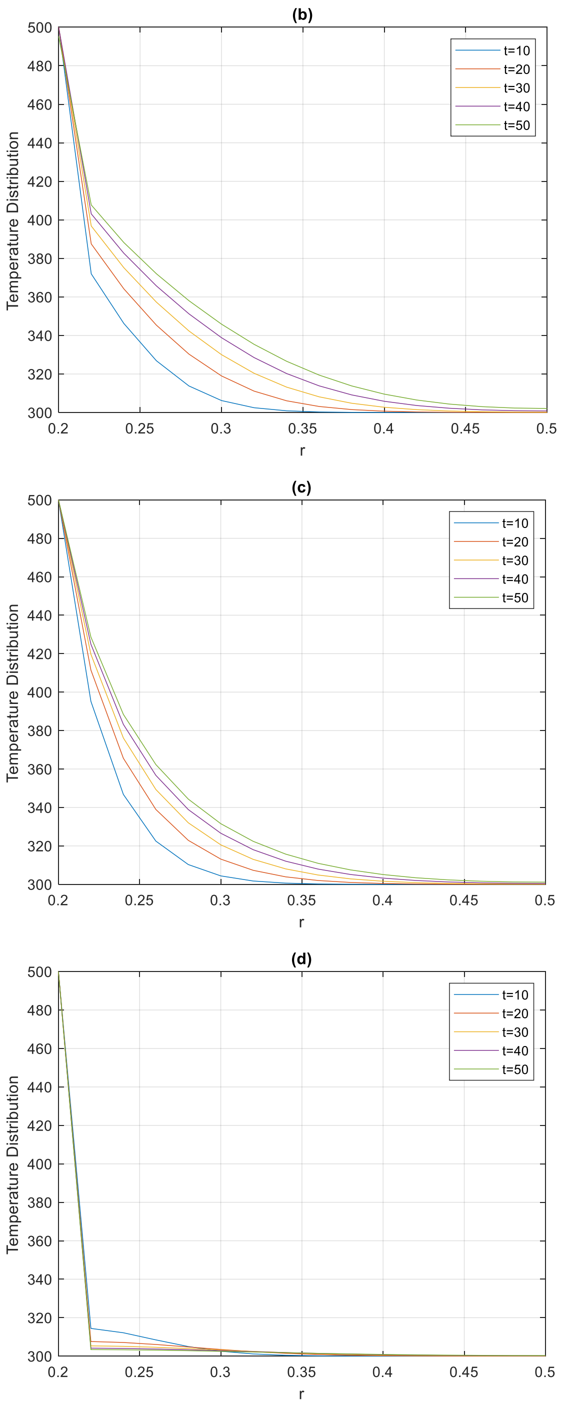


Fig.2 Temperature distribution $T(K)$ for (a) $\beta = 0, \gamma = 0$, (b) $\beta = 0.01, \gamma = 0$, (c) $\beta = 0, \gamma = 0.01$, (d) $\beta = 0.01, \gamma = 0.01$

One can observe the following from the surface plots,

1. The temperature distribution of the fin decreases as the distance from the base of the fin increases.
2. The temperature is decreased for the case when the thermal conductivity is temperature dependent.
3. The convection due to dissipation can be observed at the upper surface of the fin.
4. The difference in the temperature distribution when the thermal properties are constant and temperature dependent is observed.

V. CONCLUSIONS

In this manuscript, the attempt has been made to discuss role of temperature dependent thermal properties in heat transfer analysis. Due to consideration of temperature dependent thermal properties, the mathematical formulation of physical application in the form of boundary value problem is highly non-linear. To find mathematical solution of non-linear boundary value problem, the finite difference scheme has been proposed for governing non-linear partial differential equations. The convergence and stability analysis of finite difference solution has been done by fundamental theorems of Numerical analysis. The results obtained for heat transfer and thermal stress analysis has been validated by equilibrium and compatibility equations in classical thermoelasticity. The comparison is made for temperature, displacement and thermal stresses at each nodal point when the thermal properties are independent ($\beta = 0, \gamma = 0$) and dependent ($\beta \neq 0, \gamma \neq 0$) on temperature.

One can summaries that, the temperature dependent thermal properties plays important role in heat transfer and thermal stress analysis, particularly when solid is subjected to large temperature variation. This work gives better outline for the solution of non-linear boundary value problem. Any special case of particular interest can be derived by this approach.

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