A SIMPLE REDUCTION OF THE WALLIS' DOUBLE INEQUALITY

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ABSTRACT

In this article, a simple reduction of the Wallis' double inequality is presented.

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1. Introduction

In this article, we use $\mathbb{N}$ to denote the set of all positive integers and

$$n!! := \prod_{i=0}^{[(n-1)/2]} (n-2i),\ n \in \mathbb{N}. \tag{1}$$

Here in (1) the symbol $[t]$ denotes the largest integer less than or equal to the real number t.

$$W_n := \frac{(2n-1)!!}{(2n)!!},\ n \in \mathbb{N}, \tag{2}$$

which is called the Wallis' ratio in the literature.

The Euler gamma function is defined and denoted for $\operatorname{Re} z > 0$ by

$$\Gamma(z) := \int_0^\infty t^{z-1} e^{-t} dt. \tag{3}$$

For the quantity of $n!$, an important result is the following Stirling's formula

$$n! \sim \frac{\sqrt{2\pi n} n^n}{e^n},\ n \to \infty, \tag{4}$$

which is a useful tool in many branches of mathematics.

Consider the Wallis' ratio W_n defined by (2). This quantity is interesting in the probability theory-for example (see [4, Chapter III]), the two events, (a) the first return to the origin takes place at time $2n$, (b) the first passage through -1 occurs at time $2n-1$, have the common probability $\frac{W_n}{2n-1}$.

The following two-sided inequality

$$\frac{1}{2}\frac{1}{\sqrt{n}} \le W_n < \frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{n}},\ n \in \mathbb{N}, \tag{5}$$

which is called ([8, p.103]) Wallis' double inequality, is a simple and important form for bounding the Wallis' ratio W_n.

Some approximations for the quantity W_n were also investigated in [3, 5, 6].

One form of the Wallis' formula (see [2, p. 10], [10, p. 211]) is

$$\frac{2}{\pi} = \prod_{n=1}^{\infty} \frac{(2n-1)(2n+1)}{(2n)^2}. \tag{6}$$

Another important form of Wallis' formula, which is equivalent to (6), is [9, pp. 181-184]

$$\lim_{n\to\infty}(2n+1)W_n^2 = \frac{2}{\pi}. \tag{7}$$

There is a close relationship between the Stirling's formula and the Wallis' formula. For example, the determination of the constant $\sqrt{2\pi}$ in the proof of the Stirling's formula (4) or in the proof of the Stirling's asymptotic formula

$$\Gamma(x) \sim \frac{\sqrt{2\pi}x^{x-1/2}}{e^x},\ x \to \infty \tag{8}$$

relies on the Wallis' formula (see [2, pp. 18-20], [10, pp. 213-215], [9, pp. 181-184]).

In this article, we shall give a new, simple way to reduce the Wallis' double inequality (5). For the convenience, we restate it as follows:

Theorem 1 ([8, p. 103])**.** For all $n \in \mathbb{N}$,

$$\frac{1}{2}\frac{1}{\sqrt{n}} \le W_n < \frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{n}}. \tag{9}$$

2. New reduction of Theorem 1

Proof of Theorem 1. By Theorem 1.2 of [7], we know that the function

$$f(x) := \frac{e^x\Gamma(x+1)}{(x+\frac{1}{2})^{x+\frac{1}{2}}} \tag{10}$$

is strictly increasing from $(-\frac{1}{2},\infty)$ onto $\left(\sqrt{\frac{\pi}{e}},\sqrt{\frac{2\pi}{e}}\right)$.

Hence we obtain that

$$\frac{\sqrt{e\pi}}{2} \le \frac{e^{n-1/2}\Gamma(n+\frac{1}{2})}{n^n} < \sqrt{\frac{2\pi}{e}},\ n \in \mathbb{N} \tag{11}$$

and

$$\lim_{n\to\infty}\frac{e^{n-1/2}\Gamma(n+\frac{1}{2})}{n^n} = \sqrt{\frac{2\pi}{e}}. \tag{12}$$

We also see that the middle term in (11) as a sequence is strictly increasing and that the constants $\frac{\sqrt{e\pi}}{2}$ and $\sqrt{\frac{2\pi}{e}}$ in (11) are best possible.

Since (see [1, p. 258])

$$\Gamma\left(n+\frac{1}{2}\right)=W_n n!\sqrt{\pi},\ n\in\mathbb{N}, \tag{13}$$

where W_n is defined by (2), we can rewrite (11) as

$$\frac{e}{2}\leq\frac{W_n e^n n!}{n^n}<\sqrt{2},\ n\in\mathbb{N}, \tag{14}$$

and rewrite (12) as

$$\lim_{n\to\infty}\frac{W_n e^n n!}{n^n}=\sqrt{2}. \tag{15}$$

The constants $\frac{e}{2}$ and $\sqrt{2}$ in (14) are best possible. The middle term in (14) as a sequence is strictly increasing.

By Theorem 1.1 of [7], we know that the function

$$g(x):=\frac{x^{x+\frac{1}{2}}}{e^x\Gamma(x+1)} \tag{16}$$

is strictly increasing from $(0,\infty)$ onto $(0,\frac{1}{\sqrt{2\pi}})$.

Therefore we have

$$\frac{1}{e}\leq\frac{n^{n+1/2}}{e^n n!}<\frac{1}{\sqrt{2\pi}},\ n\in\mathbb{N} \tag{17}$$

and

$$\lim_{n\to\infty}\frac{n^{n+1/2}}{e^n n!}=\frac{1}{\sqrt{2\pi}}. \tag{18}$$

Note that we also use the fact that

$$\Gamma(n+1)=n!$$

in order to get (17) and (18). We also see that the middle term in (17) as a sequence is strictly increasing and that the lower bound $\frac{1}{e}$ and the upper bound $\frac{1}{\sqrt{2\pi}}$ in (17) are best possible.

From (14) and (17), we obtain that

$$\frac{1}{2}\leq W_n\sqrt{n}<\frac{1}{\sqrt{\pi}},\ n\in\mathbb{N}, \tag{19}$$

which is equivalent to (9), and that the sequence $\{W_n\sqrt{n}\}_{n=1}^{\infty}$ is strictly increasing. The constants $\frac{1}{2}$ and $\frac{1}{\sqrt{\pi}}$ are best possible in (19), therefore they are also best in (9).

The proof of Theorem 1 is thus completed. □

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